Exponential Function on the Complex Plane

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Abstract

In this article we explore the definition of a transcendent function on the complex plane. How did mathematicians arrive at this definition? The answer, as we will see by solving a simple differential equation, is by infinite series, by which the functions like e^x, $\sin x$, $\cos x$ are defined.

How are we to make sense of numbers like:

$$e^{2+3i}$$

Certainly, there must be a way to define and calculate such numbers. Let's try to trace back the mathematical thinking which led to such results. The problem with which our exploration might begin is the following: we must find a function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(x), f(0) = 1$. Let's try to search for the solution in the set of all polynomials:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

The derivation is now:

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots$$

But $f'(x) = f(x)$ which means that:

$$a_1 + 2a_2x + 3a_3x^2 + \cdots = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

We now use the thoeorem of equality between two polynomials:

$$a_1 = a_0$$
$$2a_2 = a_1$$
$$3a_3 = a_2$$

and eventually:

$$(n + 1)a_{n+1} = a_n$$

We solve the equations and see that:

$$a_{n+1} = \frac{1}{(n+1)n!}a_0$$

But, $f(0) = 1$ which means that:

$$a_n = \frac{1}{n!}$$

Because $a_n \neq 0$ for every $n \in \mathbb{N}$, the solution is not in the set of all polynomials. But the sequence:

$$1, 1 + \frac{x}{1!}, 1 + \frac{x}{1!} + \frac{x^2}{2!}, \cdots, 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}, \cdots$$

approximates the solution to our differential equation. It makes sense to consider the limit of the sequence as the solution:

$$f(x) = \lim_{n\to\infty}\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right)$$

Which we write short as:

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and call the expression a *series*. From elementary calculus we know that $(e^x)' = e^x$ and therefore:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

This defines the exponential function not only in the real plane, but also in the complex plane. We return to the beginning of the article and conclude that our number is calculated by the formula:

$$e^{2+3i} = \sum_{n=0}^{\infty} \frac{(2+3i)^n}{n!}$$

One of its approximations for $n = 2$ would be:

$$e^{2+3i} \approx \frac{1}{2} - 3i$$

In a similar way, using other theorems of analysis, we conclude that:

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$$